Convergence of Lagrangian mean curvature flow in Kähler-Einstein manifolds

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1 Introduction

The problem on Lagrangian submanifolds in Calabi-Yau manifolds or general Kähler manifolds has been the subject of intense study over the last few decades. They are important both in mathematics and physics because minimal Lagrangian submanifolds in Calabi-Yau manifolds are related to T-duality and Mirror symmetry in physics in the fundamental paper [23]. However, to construct a minimal Lagrangian submanifold is very difficult. Here we will use Lagrangian mean curvature flow to give some sufficient conditions for the existence of minimal Lagrangian submanifolds in a general Kähler-Einstein manifold.

A mean curvature flow is called Lagrangian mean curvature flow if the initial submanifold is Lagrangian. It is proved by [18] that the property of Lagrangian is preserved along the mean

curvature flow. Thus, it is possible to use the flow method to construct minimal Lagrangian submanifolds. A natural question is how to analyze the long time behavior or singularities along the Lagrangian mean curvature flow. In [24], Thomas-Yau conjectured that under some stability conditions the Lagrangian mean curvature flow exists for all time and converges to a special Lagrangian submanifold in its hamiltonian deformation class. There are several results relevant to this conjecture. In [20][22] Smoczyk and Smoczyk-Wang proved the long time existence and convergence of the Lagrangian mean curvature flow into a flat space under some convexity conditions respectively, and in [1] Chau-Chen-He studied the flow of entire Lagrangian graph with Lipschitz continuous initial data. In [26], M. T. Wang also proved the convergence for the graph of a symplectomorphism between Riemann surfaces. However, the flow will develop finite time singularities in general, and the readers are referred to [25] [6][12][13][10] and references therein.

In this paper, we will consider the Lagrangian mean curvature flow in a general Kähler-Einstein manifold with arbitrary dimension under some stability conditions. Let $(M, \bar{g})$ be a complete Kähler-Einstein manifold with scalar curvature $\bar{R}$ and

$$K_5 = \sum_{i=0}^{5} \sup_{\bar{M}} |\bar{\nabla}^i \bar{R}m| < \infty, \quad inj(M) \geq \iota_0 > 0, \tag{1.1}$$

where $inj(M)$ is the injectivity radius of $(M, \bar{g})$. The first main result is

Theorem 1.1. *Let $(M, \bar{g})$ be a complete Kähler-Einstein manifold satisfying (1.1) with scalar curvature $\bar{R} < 0$, and L be a compact Lagrangian submanifold smoothly immersed in M. For any $V_0, \Lambda_0 > 0$, there exists $\epsilon_0 = \epsilon_0(V_0, \Lambda_0, \bar{R}, K_5, \iota_0) > 0$ such that if L satisfies*

$$\mathrm{Vol}(L) \leq V_0, \quad |A| \leq \Lambda_0, \quad \int_L |H|^2 \leq \epsilon_0, \tag{1.2}$$

where A is the second fundamental form of L in M and H is the mean curvature vector, then the Lagrangian mean curvature flow with the initial data L will converge exponentially fast to a minimal Lagrangian submanifold in M.

Here we need to assume the scalar curvature of the ambient Kähler-Einstein manifold is negative because any minimal Lagrangian submanifold is strictly stable in this situation. Thus, it is natural to expect that for any small perturbation of a minimal Lagrangian submanifold, the Lagrangian mean curvature flow will exist for all time and deform it to a minimal Lagrangian submanifold. Theorem 1.1 shows that this is indeed true, but we don't need to assume the existence of minimal Lagrangian submanifolds.

For a Kähler-Einstein manifold with nonnegative scalar curvature, we have the result:

Theorem 1.2. *Let $(M, \bar{g})$ be a complete Kähler-Einstein manifold satisfying (1.1) with scalar curvature $\bar{R} \geq 0$, and L be a compact Lagrangian submanifold smoothly immersed in M. For any $V_0, \Lambda_0, \delta_0 > 0$, there exists $\epsilon_0 = \epsilon_0(V_0, \Lambda_0, \delta_0, \bar{R}, K_5, \iota_0) > 0$ such that if*

1. *the mean curvature form of L is exact,*

2. L *satisfies*

$$\lambda_1 \geq \frac{\bar{R}}{2n} + \delta_0, \quad \mathrm{Vol}(V) \leq V_0, \quad |A| \leq \Lambda_0, \quad \int_L |H|^2 \leq \epsilon_0,$$

where λ_1 *is the first eigenvalue of the Laplacian operator with respect to the induced metric on* L,

then the Lagrangian mean curvature flow with the initial data L *will converge exponentially fast to a minimal Lagrangian submanifold in* M.

For Lagrangian submanifolds in Kähler-Einstein manifolds with positive scalar curvature, a notion of hamiltonian stability was introduced in [14] to characterize the variations of the submanifold under hamiltonian deformations. The hamiltonian stability is more natural than the standard stability for the case when the scalar curvature is positive. For example, $\mathbb{RP}^n$ and the Clifford torus $\mathbb{T}^n$ in $\mathbb{CP}^n$ are hamiltonian stable but not (Lagrangian) stable in the standard sense. Thus, to get a convergence result for the Lagrangian mean curvature flow it is natural to expect that the deformation along the flow is hamiltonian, which is equivalent to say that the mean curvature form along the flow is exact. Fortunately, the exactness of the mean curvature form is preserved along the mean curvature flow. This is why we need the assumption 1 in Theorem 1.2.

In [14], Y. G. Oh proved that a minimal Lagrangian submanifold is hamiltonian stable if and only if the first eigenvalue of the Laplacian operator $\lambda_1 \geq \bar{R}/2n$. Thus, the assumption 3 of Theorem 1.2 on the first eigenvalue ensures that the limit minimal Lagrangian submanifold is strictly hamiltonian stable. Since for the well-known examples $\mathbb{RP}^n$ and the Clifford torus $\mathbb{T}^n$ in $\mathbb{CP}^n$ the first eigenvalue $\lambda_1 = \bar{R}/2n$, we can see that Theorem 1.2 cannot be applied. It is interesting to know whether we have the corresponding result in this situation. This phenomenon is similar to the Kähler-Ricci flow on Kähler manifolds with nonzero holomorphic vector fields (cf. [3]).

Before stating the third result, we introduce

Definition 1.3. *A vector field* X *is called an essential hamiltonian variation of* L *, if* X *can be written as* $X = J\nabla f$ *where* $f \notin E_{\lambda_1}$. *Here* E_{λ_1} *is the first eigenspace of the Laplacian operator* Δ *on* L.

Theorem 1.4. *Let* $(M, \bar{g})$ *be a compact Kähler-Einstein manifold with* $\bar{R} \geq 0$. *Suppose that* $\phi : L \to M$ *is a compact minimal Lagrangian submanifold with the first eigenvalue* $\bar{R}/2n$ *and* X *is an essential hamiltonian variation of* $L_0 = \phi(L)$. *Let* $\phi_s : L \to M(s \in (-\eta, \eta))$ *with* $\phi_0 = \phi$ *be a one-parameter family of hamiltonian deformations generated by* X. *Then there exists* $\epsilon_0 = \epsilon_0(X, L_0, M) > 0$ *such that if* $L_s = \phi_s(L) \subset M$ *satisfying*

$$\|\phi_s - \phi_0\|_{C^3} \leq \epsilon_0,$$

then the Lagrangian mean curvature flow with the initial Lagrangian submanifold L_s *will converge exponentially fast to a minimal Lagrangian submanifold in* M.

Note that we can show that a minimal Lagrangian submanifold L with $\lambda_1 = \bar{R}/2n$ is strictly hamiltonian stable along an essential hamiltonian variation X(cf. Lemma 6.4). Theorem 1.4 says that the flow will exist for all time and converge if the initial Lagrangian submanifold is a small perturbation of L along essential hamiltonian variations, which is reasonable since L is strictly hamiltonian stable along these directions.

The idea of the proofs of Theorem 1.1 and 1.2 is similar to that used in [3]. First, we use the smallness of the mean curvature vector in a short time interval to get the exponential decay of the L^2 norm of the mean curvature vector, which is a crucial step in the whole argument. Then, by a simple observation(cf. Lemma 3.4) we can get all higher order estimate of the second fundamental form from the decay of the L^2 norm of the mean curvature. This step relies on the noncollapsing assumption of Lagrangian submanifolds, which is a technical condition and can be removed in the proof of the main theorems. This step is different from the Kähler-Ricci flow in [3][4], where we use the parabolic Moser iteration to get C^0 order estimate of the Kähler potential. Then, we can show that the exponential decay of the mean curvature vector implies that the second fundamental form is uniformly bounded for any time interval and we can extend the solution for all time. The readers are referred to [3] for more details of the argument.

In a forthcoming paper, we expect to extend the argument in the present paper to the case when the initial submanifold is not Lagrangian. Our argument might be also useful for the symplectic mean curvature flow (cf. [5][9]), and we will explore this in the future.

This paper is organized as follows: In Section 2, we recall some basic facts and evolution equations of mean curvature flow and Lagrangian submanifolds. In particular, we will give some details of the proof which will be used in the paper. In Section 3, we will show several technical lemmas along the Lagrangian mean curvature flow. In Section 4 and 5, we will finish the proof of Theorem 1.1 and 1.2. In Section 6, we will recall some basic facts on the deformation of minimal Lagrangian submanifolds and finish the proof of Theorem 1.4. In the last section, we collect some examples where our theorems can be applied.

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2 Notations and preliminaries

In this section, we recall some evolution equations from [5] for the mean curvature flow in arbitrary dimension and codimension, and then we discuss the special case of Lagrangian mean curvature flow.

Let $(M, \bar{g})$ be a m-dimensional Riemannian manifold and $F_0 : L \to M$ be a smoothly immersed submanifold with dimension n. We consider the a one-parameter family of smooth maps $F_t : L \to M$ with the image $L_t = F_t(L)$ smooth submanifold in M and F satisfies

$$\frac{\partial}{\partial t} F(x,t) = H(x,t), \quad F(x,0) = F_0(x). \tag{2.1}$$

Here $H(x,t)$ is the mean curvature vector of L_t at $F(x,t)$ in M. Choose a local orthonormal frame $e_1, \cdots, e_n, e_{n+1}, \cdots, e_m$ of M along L_t such that $e_1, \cdots, e_n$ are tangent vectors of L_t

and $e_{n+1}, \cdots, e_m$ are in the normal bundle over L_t. The second fundamental form and the mean curvature operator are given by

$$A = A^\alpha e_\alpha, \quad H = -H^\alpha e_\alpha,$$

where $\alpha \in \{n+1, \cdots, m\}$. Let $A^\alpha = (h^\alpha_{ij})$ where (h^α_{ij}) is a matrix given by

$$h^\alpha_{ij} = \bar{g}(\bar{\nabla}_{e_i} e_\alpha, e_j) = \bar{g}(\bar{\nabla}_{e_j} e_\alpha, e_i) = h^\alpha_{ji}$$

where $\bar{\nabla}$ is the Levi-Civita connection on M. The mean curvature $H^\alpha = g^{ij} h^\alpha_{ij}$, where $g_{ij} = \bar{g}(e_i, e_j)$ is the induced metric on L. By direct calculation we have the evolution equation of the induced metric

$$\frac{\partial}{\partial t} g_{ij} = -2H^\alpha h^\alpha_{ij}.$$

With these notations, we have the evolution equations of the second fundamental form and the mean curvature vector.

Lemma 2.1. *(cf. [5]) The evolution equation of the second fundamental form is given by*

$$\begin{aligned}
\frac{\partial}{\partial t} h^\alpha_{ij} &= \nabla_i \nabla_j H^\alpha - H^\gamma h^\gamma_{jl} h^\alpha_{il} + h^\beta_{ij} b^\alpha_\beta + H^\beta \bar{R}_{\alpha j \beta i} && (2.2)\\
&= \Delta h^\alpha_{ij} + h^\beta_{il} h^\beta_{ml} h^\alpha_{mj} - H^\beta (h^\beta_{mi} h^\alpha_{mj} + h^\beta_{mj} h^\alpha_{mi}) + h^\beta_{ij} h^\beta_{ml} h^\alpha_{ml} - h^\beta_{im} h^\beta_{jl} h^\alpha_{ml} \\
&\quad -(\bar{\nabla}_l \bar{R}_{\alpha jil} + \bar{\nabla}_i \bar{R}_{\alpha ljl}) - (\bar{R}_{\beta\alpha jl} h^\beta_{il} + \bar{R}_{\alpha\beta il} h^\beta_{jl}) + (\bar{R}_{mllj} h^\alpha_{im} + \bar{R}_{illm} h^\alpha_{jm}) \\
&\quad +2\bar{R}_{iljm} h^\alpha_{ml} + \bar{R}_{\alpha l\beta l} h^\beta_{ij} + h^\beta_{ij} b^\alpha_\beta, && (2.3)
\end{aligned}$$

where $b^\beta_\alpha = \bar{g}(\frac{\partial}{\partial t} e_\alpha, e_\beta)$. The equation of mean curvature vector is given by

$$\frac{\partial}{\partial t} H^\alpha = \Delta H^\alpha + H^\beta h^\beta_{lm} h^\alpha_{ml} + H^\beta \bar{R}_{\alpha l \beta l} + H^\beta b^\alpha_\beta. \tag{2.4}$$

Here $\bar{R}_{ABCD}$ is the curvature tensor in M and we choose the convention such that $\bar{R}_{uvuv} > 0$ for round spheres.

Proof. The equations (2.2)(2.3) follow directly from Lemma 2.3-2.5 and Proposition 2.6 in [5], and (2.4) follows from (2.2) and the definition of H.

□

Now we recall some basic facts of Lagrangian mean curvature flow from [18]-[21]. Assume that $(M, \bar{g}, J)$ is a Kähler-Einstein manifold of real dimension $m = 2n$, and L is an n-dimensional manifold smoothly immersed into M by a smooth map $F : L \to M$. Let $\bar{\omega}$ be the associate Kähler form of the metric $\bar{g}$. The submanifold $L_0 = F(L) \subset M$ is called Lagrangian if

$$F^* \bar{\omega} = 0, \quad \text{on } L.$$

Choose normal coordinates $\{x^i\}$ for L and we have that $e_i = \partial_i F$ are the tangent vectors of L. Since L is Lagrangian, Je_i is a normal vector for any $i = 1, \cdots, n$. In fact,

$$\bar{g}(Je_i, e_j) = \bar{\omega}(e_i, e_j) = 0.$$

Hence, $\{e_i, Je_i\}$ is a local coordinate frame of M. For convenience, we use that an underlined index denotes the application of the complex structure J. For example

$$\bar{g}_{i\underline{j}} = \bar{g}(e_i, Je_j).$$

For simplicity, we denote by h_{ij}^k the second fundamental form $h_{ij}^{\underline{k}} = -\bar{g}(Je_k, \bar{\nabla}_{e_i} e_j)$. Since L is Lagrangian, it is easy to check that the second fundamental form has full symmetry

$$h_{ij}^k = h_{ji}^k = h_{kj}^i.$$

The mean curvature vector $H = -H^i Je_i$ where $H^i = g^{kl} h_{kl}^i$. The norms of the second fundamental form and the mean curvature vector are given by

$$|A|^2 = h_{ij}^k h_{pq}^l g^{ip} g^{jq} g_{kl}, \quad |H|^2 = H^i H^j g_{ij}.$$

We define the mean curvature form by $\alpha_H = g_{ij} H^j dx^i$, and we have the following well-known result:

Lemma 2.2. *If M is a Kähler-Einstein manifold, then α_H is a closed 1-form.*

Proof. By the full symmetry of h_{ij}^k, we have

$$d\alpha_H(e_i, e_j) = \nabla_i H^j - \nabla_j H^i = \nabla_i h_{kj}^k - \nabla_j h_{ki}^k = \bar{R}_{jik\underline{k}},$$

where we have used the Codazzi equation

$$\nabla_i h_{jk}^l - \nabla_j h_{ik}^l = -\bar{R}_{\underline{l}kji}.$$

Since M is a Kähler-Einstein manifold, by equality (2.10) below we have

$$d\alpha_H(e_i, e_j) = \bar{R}_{\underline{j}i} = \frac{\bar{R}}{2n} \bar{\omega}(e_j, e_i) = 0, \quad \text{on } L,$$

since L is Lagrangian. The lemma is proved. $\square$

For the mean curvature flow (2.1), if the initial data L_0 is Lagrangian, then the submanifolds L_t are all Lagrangian (cf. [18]). Thus, we call the flow (2.1) the Lagrangian mean curvature flow if the initial submanifold L_0 is Lagrangian. It was proved by [21] that the exactness of the mean curvature form of L_t is preserved along the Lagrangian mean curvature flow.

Lemma 2.3. *Along the Lagrangian mean curvature flow, we have*

$$\frac{\partial}{\partial t} H^i = \Delta H^i + H^j h_{lm}^j h_{ml}^i + H^j \bar{R}_{\underline{i}l\underline{j}l} - H^j H^k h_{kj}^i. \tag{2.5}$$

The second fundamental form satisfies

$$\frac{\partial}{\partial t} h_{ij}^k = \nabla_i \nabla_j H^k - H^m h_{jl}^m h_{il}^k - H^l h_{mk}^l h_{ij}^m + H^m \bar{R}_{\underline{k}j\underline{m}i}. \tag{2.6}$$

Proof. Since L_t is Lagrangian along the flow, we have

$$b^{\underline{i}}_{\underline{j}} = \bar{g}(\frac{\partial}{\partial t}e_{\underline{i}}, e_{\underline{j}}) = \bar{g}(J\nabla_{e_i}(-H^k e_{\underline{k}}), e_{\underline{j}}) = -H^k h^k_{ij}. \tag{2.7}$$

By (2.4), we have

$$\frac{\partial}{\partial t}H^i = \Delta H^i + H^j h^j_{lm} h^i_{ml} + H^j \bar{R}_{\underline{i}l\underline{j}l} - H^j H^k h^k_{ij}.$$

The equation (2.6) follows directly from (2.7) and Lemma 2.1.

$\square$

Lemma 2.4. *(cf. [18]) If the initial mean curvature form is exact, then there exists a smooth angle function $\theta(x,t)$ such that $\alpha_H = d\theta$ and*

$$\frac{\partial\theta}{\partial t} = \Delta\theta + \frac{\bar{R}}{2n}\theta. \tag{2.8}$$

Proof. It follows from [18] that $\alpha_H(t)$ are exact as long as the solution exists if the initial mean curvature form is exact. Since $H^i = \nabla^i\theta$, and we calculate

$$\begin{aligned}
\Delta\nabla^i\theta &= \nabla_k\nabla_i\nabla^k\theta = (\nabla_i\nabla_k\nabla^k\theta + R_{kikl}\nabla^l\theta) \\
&= \nabla^i\Delta\theta + (\bar{R}_{kikl} + H^m h^m_{il} - h^m_{kl}h^m_{ki})\nabla^l\theta.
\end{aligned} \tag{2.9}$$

Combining (2.9) with (2.5), we have

$$\begin{aligned}
\nabla^i\frac{\partial\theta}{\partial t} &= \nabla^i\Delta\theta + (\bar{R}_{\underline{i}m\underline{l}m} + \bar{R}_{miml})\nabla^l\theta \\
&= \nabla^i\Delta\theta + \frac{\bar{R}}{2n}\nabla^i\theta.
\end{aligned}$$

Hence, (2.8) is proved.

$\square$

For the readers'convenience, we collect some basic facts on curvatures in a Kähler manifold. Let $(M, \bar{g})$ be a Kähler manifold, the Ricci curvature is given by

$$\bar{R}_{AC} = \bar{g}^{BD}\bar{R}_{ABCD}.$$

Here doubled latin capitals are summed from 1 to $2n$. Now in the local frame $\{e_i, Je_i\}$ we calculate

$$\begin{aligned}
\bar{R}_{AB} &= \bar{R}(e_A, e_k, e_B, e_k) + \bar{R}(e_A, Je_k, e_B, Je_k) \\
&= \bar{R}(e_A, e_k, Je_B, Je_k) - \bar{R}(e_A, Je_k, Je_B, e_k) \\
&= \bar{R}(e_k, e_A, Je_k, Je_B) + \bar{R}(e_A, Je_k, e_k, Je_B) \\
&= -\bar{R}(Je_k, e_k, e_A, Je_B) = \bar{R}_{A\underline{B}k\underline{k}}.
\end{aligned}$$

Hence, we have

$$\bar{R}_{AB} = \bar{R}_{A\underline{B}k\underline{k}}, \quad \bar{R}_{A\underline{B}} = -\bar{R}_{ABk\underline{k}}. \tag{2.10}$$

The scalar curvature $\bar{R} = \bar{R}_{kk} + \bar{R}_{\underline{kk}} = 2\bar{R}_{kk}$. Since $\bar{g}$ is a Kähler-Einstein metric, we have $\bar{R}_{ij} = \frac{\bar{R}}{2n}\bar{g}_{ij}$.

3 Estimates

In this section, we derive some estimates along Lagrangian mean curvature flow.

3.1 The mean curvature vector

In this subsection, we will prove that the L^2 norm of the mean curvature vector decays exponentially under certain conditions. More precisely, we will prove that the L^2 norm of the mean curvature vector will decays exponentially when the mean curvature is small and the scalar curvature of the ambient Kähler-Einstein manifold is negative. For the case of nonnegative scalar curvature, an interesting condition on the exactness of the mean curvature form is assumed to ensure the exponential decay when the mean curvature is small.

Lemma 3.1. *Let $(M, \bar{g})$ be a Kähler-Einstein manifold with scalar curvature $\bar{R}$. For any $\Lambda, \epsilon > 0$, if the solution $L_t(t \in [0,T])$ of Lagrangian mean curvature flow satisfies*

$$|A|(t) \leq \Lambda, \quad |H|(t) \leq \epsilon, \quad t \in [0,T],$$

then we have the inequality

$$\frac{d}{dt}\int_{L_t} |H|^2 d\mu_t \leq \Big(\frac{1}{n}\bar{R} + 2\Lambda\epsilon\Big)\int_{L_t} |H|^2 d\mu_t, \quad t \in [0,T]. \tag{3.1}$$

Moreover, if we assume that the mean curvature form of L_0 is exact, then

$$\frac{\partial}{\partial t}\int_{L_t} |H|^2 d\mu_t \ \leq \ -2\Big(\lambda_1 - \frac{\bar{R}}{2n} - \Lambda\epsilon\Big)\int_{L_t} |H|^2 d\mu_t, \tag{3.2}$$

where λ_1 is the first eigenvalue of Δ with respect to the induced metric on L.

Proof. By (2.5) we calculate

$$\begin{aligned}
& \frac{\partial}{\partial t}\int_L |H|^2 d\mu_t \\
= & \int_{L_t} \frac{\partial g_{ij}}{\partial t} H^i H^j + 2H^i \frac{\partial}{\partial t} H^i - |H|^4 \\
= & \int_L 2H^i \Delta H^i + 2H^i H^j h^j_{kl} h^i_{lk} + 2H^i H^j \bar{R}_{\underline{i}m\underline{j}m} - 4H^i H^j H^k h^i_{jk} - |H|^4 \\
\leq & \int_L -2|\nabla_k H^i|^2 + 2H^i H^j h^j_{kl} h^i_{lk} + 2H^i H^j \bar{R}_{\underline{i}m\underline{j}m} - 4H^i H^j H^k h^i_{jk} - |H|^4
\end{aligned} \tag{3.3}$$

We claim that for any vector field $X = X^i e_i$ on L, the inequality holds

$$\int_L |\nabla_i X^k|^2 - h^l_{km} h^i_{km} X^i X^l \geq \int_L |\nabla_i X^i|^2 - H^m h^m_{il} X^i X^l - \bar{R}_{kikl} X^i X^l. \tag{3.4}$$

In fact,

$$
\begin{aligned}
0 &\le \frac{1}{2}\sum_{i,k}\int_L |\nabla_i X^k - \nabla_k X^i|^2 = \sum_{i,k}\int_L |\nabla_i X^k|^2 - g(\nabla_i X^k, \nabla_k X^i) \\
&= \sum_{i,k}\int_L |\nabla_i X^k|^2 + g(\nabla_k\nabla_i X^k, X^i). \qquad (3.5)
\end{aligned}
$$

Note that we can change the covariant derivatives

$$
\begin{aligned}
\nabla_k\nabla_i X^k &= \nabla_i\nabla_k X^k + R_{kikl}X^l \\
&= \nabla_i\nabla_k X^k + (\bar{R}_{kikl} + h^m_{kk}h^m_{il} - h^m_{kl}h^m_{ik})X^l, \qquad (3.6)
\end{aligned}
$$

where R_{ijkl} is the curvature tensor on L and we used the Gauss equation

$$
R_{ijkl} = \bar{R}_{ijkl} + h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}.
$$

Thus, (3.5) and (3.6) imply that

$$
0 \le \int_L |\nabla_i X^k|^2 - |\nabla_i X^i|^2 + \bar{R}_{kikl}X^iX^l + H^m h^m_{il}X^iX^l - h^l_{km}h^i_{km}X^iX^l,
$$

which proves (3.4).

Now we apply the inequality (3.4) for the vector H^ie_i and combine this with (3.3)

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_{L_t} |H|^2 d\mu_t \\
\le &\int_{L_t} -2|\nabla_k H^i|^2 + 2H^iH^jh^j_{kl}h^i_{lk} + 2H^iH^j\bar{R}_{\underline{i}m\underline{j}m} - 4H^iH^jH^kh^i_{jk} - |H|^4 \\
\le &\int_{L_t} -2|\nabla_i H^i|^2 + 2(\bar{R}_{kikl}H^iH^l + H^iH^j\bar{R}_{\underline{i}m\underline{j}m}) - 2H^iH^jH^kh^i_{jk} - |H|^4. \qquad (3.7)
\end{aligned}
$$

Note that $\bar{R}_{kikl} + \bar{R}_{\bar{i}k\bar{l}k} = \bar{R}_{il}$, by the assumption we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\int_{L_t} |H|^2 d\mu_t &\le \int_{L_t} 2\bar{R}_{ij}H^iH^j - 2H^iH^jH^kh^i_{jk} \\
&\le \Big(\frac{\bar{R}}{n} + 2\Lambda\epsilon\Big)\int_{L_t} |H|^2 d\mu_t.
\end{aligned}
$$

Thus, (3.1) is proved.

If we assume that the mean curvature form of L_0 is exact, then by Lemma 2.4 the mean curvature form is also exact for all t. Thus, there exists a smooth function $\theta(x,t)$ with $H^i = \nabla^i\theta$, which implies

$$
\int_{L_t} |\nabla_i H^i|^2 = \int_{L_t} |\Delta\theta|^2 \ge \lambda_1 \int_{L_t} |\nabla\theta|^2 = \lambda_1 \int_{L_t} |H|^2.
$$

Combining this with (3.7), we have

$$
\frac{\partial}{\partial t}\int_{L_t} |H|^2 d\mu_t \le -2\Big(\lambda_1 - \frac{\bar{R}}{2n} - \Lambda\epsilon\Big)\int_{L_t} |H|^2 d\mu_t.
$$

Thus, (3.2) is proved. $\square$

3.2 The first eigenvalue

In previous section, we know that when the scalar curvature of the ambient manifold is nonnegative, the exponential decay of the L^2 norm of mean curvature vector will depend on the behavior of the first eigenvalue of the Laplacian along the flow. In this subsection, we give some estimates on the first eigenvalue, which essentially says that the first eigenvalue will have a positive lower bound if the mean curvature vector decays exponentially.

Lemma 3.2. *Along the Lagrangian mean curvature flow, we have*

1. *For any constants $\delta, \Lambda > 0$, there exists $t_0 = t_0(n, \Lambda, K_2, \delta)$ such that if the solution L_t satisfies $|A| \le \Lambda$ for $t \in [0, t_0]$, then*

$$\sqrt{\lambda_1(t)} \ge \sqrt{\lambda_1(0)}(1-\delta) - \delta, \quad t \in [0, t_0]. \tag{3.8}$$

2. *For any constants $T, \epsilon, \gamma, \Lambda > 0$, if the solution L_t satisfies*

$$|A| \le \Lambda, \quad |\nabla H| + |H| \le \epsilon e^{-\gamma t}, \quad t \in [0, T], \tag{3.9}$$

then we have the estimate

$$\sqrt{\lambda_1(t)} \ge \sqrt{\lambda_1(0)} e^{-\frac{1}{2\gamma}(2\Lambda\epsilon + \epsilon^2)} - \frac{(K_0 + \Lambda)\epsilon}{\gamma}, \quad t \in [0, T]. \tag{3.10}$$

Proof. Let $f(x,t)$ be a eigenfunction of the Laplacian operator with respect to the induced metric on L_t satisfying

$$-\Delta f = \lambda_1(t) f, \quad \int_{L_t} f^2 d\mu_t = 1.$$

Taking derivative with respect to t, we have

$$\int_{L_t} 2f\frac{\partial f}{\partial t} - f^2|H|^2 = 0. \tag{3.11}$$

Observe that the first eigenvalue satisfies

$$\lambda_1(t) = \int_{L_t} |\nabla f|^2 d\mu_t.$$

Thus, we calculate

$$\begin{aligned} \frac{\partial \lambda_1}{\partial t} &= -\frac{\partial}{\partial t}\int_{L_t} f\Delta f d\mu_t \\ &= -\int_{L_t} 2\frac{\partial f}{\partial t}\Delta f + f\Big(\frac{\partial}{\partial t}\Delta\Big) f - f\Delta f|H|^2 \\ &= \int_{L_t} \lambda_1\Big(2f\frac{\partial f}{\partial t} - f^2|H|^2\Big) - 2H^i h^i_{kl} f\nabla_k\nabla_l f \\ &= \int_{L_t} -2H^i h^i_{kl} f \nabla_k \nabla_l f \\ &= \int_{L_t} 2H^i h^i_{kl}\nabla_k f\nabla_l f + 2(H^i h^i_{kl})_k f\nabla_l f, \end{aligned} \tag{3.12}$$

where we used the equality (3.11). Note that by the Codazzi equation we have

$$\nabla_k h^i_{kl} - \nabla_l h^i_{kk} = -\bar{R}_{\underline{i}klk}.$$

Combining this with (3.12), we have

$$\begin{aligned}
\frac{\partial \lambda_1}{\partial t} &= \int_{L_t} 2H^i h^i_{kl} \nabla_k f \nabla_l f + 2(H^i h^i_{kl})_k f \nabla_l f \\
&= \int_{L_t} 2H^i h^i_{kl} \nabla_k f \nabla_l f + 2(\nabla_k H^i h^i_{kl} + H^i \nabla_l H^i - H^i \bar{R}_{\underline{i}klk}) f \nabla_l f \\
&= \int_{L_t} 2H^i h^i_{kl} \nabla_k f \nabla_l f - |H|^2 (|\nabla f|^2 + f\Delta f) \\
&\quad -2H^i \bar{R}_{\underline{i}klk} f \nabla_l f + 2\nabla_k H^i h^i_{kl} f \nabla_l f. \qquad (3.13)
\end{aligned}$$

(1). Under the assumption 1, by Lemma 3.6 and 3.7 there exist positive constants $\underline{t} = \underline{t}(n, \Lambda, K_1)$ and $a_1 = a_1(n, \Lambda, K_2)$ such that

$$|\nabla H| \le \frac{a_1}{\sqrt{t}}, \quad t \in (0, \underline{t}].$$

Thus, by (3.13) we have

$$\begin{aligned}
\frac{\partial \lambda_1}{\partial t} &\ge -4\Lambda^2 \lambda_1 - 2K_0 \Lambda \lambda_1^{\frac{1}{2}} - \frac{2\Lambda a_1}{\sqrt{t}} \lambda_1^{\frac{1}{2}} \\
&= -c_1 \lambda_1 - \Big(\frac{c_2}{\sqrt{t}} + c_3\Big) \lambda_1^{\frac{1}{2}}, \quad t \in (0, \underline{t}]
\end{aligned}$$

where $c_1 = 4\Lambda^2, c_2 = 2\Lambda a_1$ and $c_3 = 2K_0\Lambda$. Thus, we have

$$\sqrt{\lambda_1(t)} \ge \sqrt{\lambda_1(0)} e^{-\frac{c_1}{2}t} - \frac{c_3}{2} t - c_2 \sqrt{t}, \quad t \in [0, \underline{t}].$$

If we choose t sufficiently small, then (3.8) is proved.

(2). Under the assumption (3.9), by (3.13) we have

$$\begin{aligned}
\frac{\partial \lambda_1}{\partial t} &\ge -2\Lambda \epsilon e^{-\gamma t} \lambda_1 - 2\epsilon^2 e^{-2\gamma t} \lambda_1 - 2K_0 \epsilon e^{-\gamma t} \lambda_1^{\frac{1}{2}} - 2\Lambda \epsilon e^{-\gamma t} \lambda_1^{\frac{1}{2}} \\
&\ge -(2\Lambda \epsilon e^{-\gamma t} + 2\epsilon^2 e^{-2\gamma t}) \lambda_1 - 2(K_0 + \Lambda) \epsilon e^{-\gamma t} \lambda_1^{\frac{1}{2}}, \quad t \in [0, T].
\end{aligned}$$

Thus, we have

$$\sqrt{\lambda_1(t)} \ge \sqrt{\lambda_1(0)} e^{-\frac{1}{2\gamma}(2\Lambda\epsilon + \epsilon^2)} - \frac{(K_0 + \Lambda)\epsilon}{\gamma}.$$

$\square$

3.3 Zero order estimates

In Section 3.1, we proved the exponential decay of the L^2 norm of the mean curvature vector under some conditions. To get a pointwise decay of the mean curvature form, we need to do more work. One way is to use the parabolic Moser iteration as in the Kähler-Ricci flow in [4] and [3]. However, the Sobolev inequality for submanifolds in a Riemannian manifold needs many restrictions (cf. [11]). Here we give a simple observation to bound the C^0 estimates by the L^2 norm. First, we introduce the following definition, which is inspired by Ricci flow [17]:

Definition 3.3. *A geodesic ball $B(p, \rho) \subset L$ is called κ-noncollapsed if $\mathrm{Vol}(B(q, s)) \geq \kappa s^n$ whenever $B(q, s) \subset B(p, \rho)$. Here the volume is with respect to the induced metric on L. A Riemannian manifold L is called κ-noncollapsed on the scale r if every geodesic ball $B(p, s)$ is κ-noncollapsed for $s \leq r$.*

Lemma 3.4. *If L_0 is κ_0-noncollapsed on the scale r_0, then for any small geodesic ball $B_t(p, \rho)$ in L_t with radius $\rho \in (0, r_0)$, we have*

$$\mathrm{Vol}(B_t(p, \rho)) \geq \kappa_0 e^{-(n+1)E(t)} \rho^n,$$

where $E(t)$ is given by

$$E(t) = \int_0^t \max_{L_s} (|A||H| + |H|^2)\, ds. \tag{3.14}$$

Proof. Recall that the evolution equation of the induced metric on L is given by

$$\frac{\partial}{\partial t} g_{ij} = -2H^k h^k_{ij},$$

which implies that the distance function satisfies

$$e^{-E(t)} d_0(x, y) \leq d_t(p, q) \leq d_0(p, q) e^{E(t)}$$

and the volume form $d\mu_t \geq e^{-E(t)} d\mu_0$, where $E(t)$ is given by (3.14). Thus, the volume of $B_t(p, \rho)$ has the estimate

$$\mathrm{Vol}(B_p(\rho)) \;=\; \int_{B_t(p,\rho)} d\mu_t \geq \int_{B_0(p, e^{-E(t)}\rho)} e^{-E(t)} d\mu_0 \geq \kappa_0 e^{-(n+1)E(t)} \rho^n,$$

as long as $\rho \leq r_0$ since L_0 is κ_0-noncollapsed on the scale r_0. The lemma is proved.

□

To derive the zero order estimate of the mean curvature vector, we prove the following simple result:

Lemma 3.5. *Suppose that L is κ-noncollapsed on the scale r. For any tensor S on L, if*

$$|\nabla S| \leq \Lambda, \qquad \int_L |S|^2\, d\mu \leq \epsilon,$$

where $\epsilon \leq r^{n+2}$, then

$$\max_L |S| \leq \Big(\frac{1}{\sqrt{\kappa}} + \Lambda\Big) \epsilon^{\frac{1}{n+2}}.$$

Proof. Assume that $|S|$ attains its maximum at point $x_0 \in L$. Thus, for any point $x \in B(x_0, \delta)$ with small $\delta > 0$ we have

$$|S(x)| \geq |S(x_0)| - \Lambda\delta > 0.$$

Hence, we have the inequality

$$\epsilon \geq \int_{B(x_0,\delta)} |S|^2 \, d\mu \geq (|S(x_0)| - \Lambda\delta)^2 \mathrm{Vol}(B(x_0, \delta)) \geq (|S(x_0)| - \Lambda\delta)^2 \kappa \delta^n.$$

Let $\delta = \epsilon^{\frac{1}{n+2}}$ and we choose ϵ small such that $\epsilon^{\frac{1}{n+2}} \leq r$, then

$$\max_L |S| \leq \Big(\frac{1}{\sqrt{\kappa}} + \Lambda\Big)\epsilon^{\frac{1}{n+2}}.$$

The lemma is proved. □

3.4 Higher order estimates

In this subsection, we collect some basic estimates for the second fundamental form, which can be proved by the maximum principle. The following result shows that the second fundamental form doesn't change too much near the initial time.

Lemma 3.6. *Along the Lagrangian mean curvature flow, if L_0 satisfies*

$$|A|(0) \leq \Lambda, \quad |H|(0) \leq \epsilon,$$

then there exists $T = T(n, \Lambda, K_1)$ such that L_t has the estimates

$$|A|(t) \leq 2\Lambda, \quad |H|(t) \leq 2\epsilon, \quad t \in [0, T]. \tag{3.15}$$

Proof. It follows from the maximum principle. Recall that by (2.3) the second fundamental form satisfies

$$\frac{\partial}{\partial t}|A| \leq \Delta|A| + c_1(n)|A|^3 + c_2(n, K_0)|A| + c_3(n, K_1).$$

Let $t_0 = \sup\{s > 0 \mid |A|(t) \leq 2\Lambda, \ t \in [0, s)\}$. Then, for $t \in [0, t_0)$ we have the inequality

$$\frac{\partial}{\partial t}|A| \leq \Delta|A| + 8\Lambda^3 c_1 + 2\Lambda c_2 + c_3, \quad t \in [0, t_0).$$

Thus, we can apply the maximum principle

$$|A|(t) \leq \max_{L_0} |A|(0) + (8\Lambda^3 c_1 + 2\Lambda c_2 + c_3)t \leq \frac{3}{2}\Lambda, \quad t \in [0, \frac{\Lambda}{2(8\Lambda^3 c_1 + 2\Lambda c_2 + c_3)}].$$

Combining this with the definition of t_0 we have

$$t_0 \geq \frac{\Lambda}{2(8\Lambda^3 c_1 + 2\Lambda c_2 + c_3)}.$$

Now we estimate the mean curvature vector. In fact, for $t \in [0, t_0]$ the mean curvature satisfies the inequality

$$\frac{\partial}{\partial t}|H| \leq \Delta|H| + |A|^2|H| + K_0|H| \leq \Delta|H| + (4\Lambda^2 + K_0)|H|,$$

which implies

$$|H|(t) \leq |H|(0)e^{(4\Lambda^2+K_0)t} \leq 2\epsilon, \quad t \in [0, \min\{t_0, \frac{\log 2}{(4\Lambda^2 + K_0)}\}].$$

Thus, (3.15) holds for

$$T = \min\{\frac{\Lambda}{2(8\Lambda^3 c_1 + 2\Lambda c_2 + c_3)}, \frac{\log 2}{(4\Lambda^2 + K_0)}\}.$$

□

For higher order estimates, K. Smoczyk proved in [19] that all higher order derivatives of the second fundamental form are bounded if the C^0 norm of A is bounded for a short time interval. However, the bound of higher order derivatives will depend on the derivatives of the second fundamental form of the initial submanifold. In this paper we need more precise estimates as in Ricci flow. The following result is taken from [7], and the readers are referred to [7] for details.

Lemma 3.7. *(cf. Theorem 3.2 in [7]) Assume that the Lagrangian mean curvature flow has a smooth solution for $t \in [0, T)$. If there is a constant Λ such that*

$$\max_{L_t} |A|^2 \leq \Lambda, \quad t \in [0, T],$$

then for any $k > 0$ there exists a constant $C_k = C_k(n, \Lambda, K_{k+1}, T)$ such that

$$\max_{L_t} |\nabla^k A|^2 \leq \frac{C_k}{t^k}, \quad t \in (0, T],$$

where $K_k = \sum_{l=0}^{k} \max_M |\bar{\nabla}^l \bar{R}m|$.

4 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. For any positive constants $\kappa, r, \Lambda, \epsilon$, we define the following subspace of Lagrangian submanifolds in M by

$$\mathcal{A}(\kappa, r, \Lambda, \epsilon) = \Big\{ L \;\Big|\; L \text{ is } \kappa\text{-noncollapsed on the scale } r \text{ with } |A|(t) \leq \Lambda,\ |H|(t) \leq \epsilon \Big\}.$$

The following result shows that the flow will have good estimates for a short time.

Lemma 4.1. *If the initial Lagrangian submanifold $L_0 \in \mathcal{A}(\kappa, r, \Lambda, \epsilon)$, then there exists $\tau = \tau(n, \Lambda, K_1) > 0$ such that $L_t \in \mathcal{A}(\frac{1}{2}\kappa, r, 2\Lambda, 2\epsilon)$ for $t \in [0, \tau]$.*

Proof. This result follows directly from Lemma 3.4 and Lemma 3.6. ☐

The following lemma is a crucial step in the whole argument of the proof. It shows that if the flow has a rough bound for a finite time interval, then we can choose some constant sufficiently small such that the mean curvature will decay exponentially and the flow has uniform bounds which are independent of the length of this time interval.

Lemma 4.2. *For any $\kappa_0, r_0, \Lambda_0, V_0, T > 0$ there exists $\epsilon_0 = \epsilon_0(\kappa_0, r_0, \Lambda_0, n, K_5, V_0) > 0$ such that if the solution $L_t (t \in [0, T])$ of the Lagrangian mean curvature flow satisfies*

1. $L_0 \in \mathcal{A}(\kappa_0, r_0, \Lambda_0, \epsilon_0)$ *and* $\mathrm{Vol}(L_0) \le V_0$,
2. $L_t \in \mathcal{A}(\frac{1}{3}\kappa_0, r_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}})(t \in [0, T])$,

Then we have the following properties

(a) The mean curvature vector satisfies

$$\max_{L_t} |H|(t) \le \epsilon_0^{\frac{1}{n+2}} e^{\frac{\bar{R}}{2n(n+2)}t}, \quad t \in [\tau, T].$$

(b) The second fundamental form

$$\max_{L_t} |A| \le 3\Lambda_0, \quad t \in [0, T].$$

(c) L_t is $\frac{2}{3}\kappa_0$-noncollapsed on the scale r_0 for $t \in [0, T]$.

Thus, the solution $L_t \in \mathcal{A}(\frac{2}{3}\kappa_0, r_0, 3\Lambda_0, \epsilon_0^{\frac{1}{n+2}})$ for $t \in [0, T]$, and by Lemma 4.1 we can extend the solution to $[0, T + \delta]$ such that $L_t \in \mathcal{A}(\frac{1}{3}\kappa_0, r_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}})(t \in [0, T + \delta])$ for some $\delta = \delta(n, \Lambda_0, K_1) > 0$.

Proof. (a). For any $\Lambda_0 > 0$, we can choose ϵ_0 small enough such that $12\Lambda_0\epsilon_0^{\frac{1}{n+2}} < -\frac{\bar{R}}{4n}$. Thus, by Lemma 3.1 the mean curvature vector satisfies

$$\int_{L_t} |H|^2 d\mu_t \le e^{\frac{\bar{R}}{2n}t} \int_{L_0} |H|^2 d\mu_0 \le V_0\epsilon_0^2 e^{\frac{\bar{R}}{2n}t}, \quad t \in [0, T]. \tag{4.1}$$

Note that $L_t \in \mathcal{A}(\frac{1}{3}\kappa_0, r_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}})$ for $t \in [0, T]$, by Lemma 3.7 there is a constant $C_1 = C_1(n, \Lambda_0, K_2)$ such that

$$|\nabla A|(t) \le C_1(n, \Lambda_0, K_2, \tau), \quad t \in [\tau, T]. \tag{4.2}$$

Here we can choose $\tau = \tau(n, \Lambda_0, K_1)$ in Lemma 4.1. Thus, by Lemma 3.5 and (4.1)(4.2) we have

$$|H|(t) \le \Big(\sqrt{\frac{3}{\kappa_0}} + C_1\Big) V_0^{\frac{1}{n+2}} \epsilon_0^{\frac{2}{n+2}} e^{\frac{\bar{R}}{2n(n+2)}t}, \quad t \in [\tau, T]. \tag{4.3}$$

where we have used the fact that L_t is $\frac{\kappa}{3}$-noncollapsed on the scale r_0 and $V_0\epsilon_0^2 \le r_0^{n+2}$ if ϵ_0 is small enough. Thus, if ϵ_0 is small such that $\Big(\sqrt{\frac{3}{\kappa_0}}+C_1\Big)V_0^{\frac{1}{n+2}}\epsilon_0^{\frac{1}{n+2}} \le 1$, then we have

$$|H|(t) \le \epsilon_0^{\frac{1}{n+2}} e^{\frac{\bar{R}}{2n(n+2)}t}, \quad t\in[\tau, T].$$

(b). By Lemma 3.7 there exist some constants $C_k = C_k(n, \Lambda_0, K_{k+1})$ such that

$$|\nabla^k A|(t) \le C_k(n, \Lambda_0, K_{k+1}, \tau), \quad t\in[\tau, T]. \tag{4.4}$$

By Lemma 3.7 and Property (a), we have

$$\int_{L_t}|\nabla^2 H|^2 d\mu_t \le \int_{L_t}|H||\nabla^4 H|d\mu_t \le V_0 C_4 \epsilon_0^{\frac{1}{n+2}} e^{\frac{\bar{R}}{2n(n+2)}t}, \quad t\in[\tau, T], \tag{4.5}$$

where we used the fact that $\mathrm{Vol}(L_t)$ is decreasing along the flow since

$$\frac{\partial}{\partial t}\mathrm{Vol}(L_t) = -\int_{L_t}|H|^2 d\mu_t \le 0.$$

Thus, by Lemma 3.5 we have

$$|\nabla^2 H| \le \Big(\sqrt{\frac{3}{\kappa_0}}+C_3\Big)C_4^{\frac{1}{n+2}}V_0^{\frac{1}{n+2}}\epsilon_0^{\frac{1}{(n+2)^2}} e^{\frac{\bar{R}}{2n(n+2)^2}t}, \quad t\in[\tau, T]. \tag{4.6}$$

Recall that by Lemma 2.3 $|A|$ satisfies the inequality

$$\frac{\partial}{\partial t}|A| \le |\nabla^2 H| + c(n)|A|^2|H| + |\bar{R}m||H|. \tag{4.7}$$

Thus, by Lemma 3.6, (4.6)(4.7) and (a) we have

$$\begin{aligned}
|A|(t) &\le |A|(\tau) + \int_\tau^t |\nabla^2 H| + (K_0+|A|^2)|H| \\
&\le 2\Lambda_0 + \Big(\sqrt{\frac{3}{\kappa_0}}+C_3\Big)C_4^{\frac{1}{n+2}}V_0^{\frac{1}{n+2}}\epsilon_0^{\frac{1}{(n+2)^2}}\frac{2n(n+2)^2}{|\bar{R}|} \\
&\quad +(K_0+36\Lambda_0^2)\epsilon_0^{\frac{1}{n+2}}\frac{2n(n+2)}{|\bar{R}|} \\
&\le 3\Lambda_0,
\end{aligned} \tag{4.8}$$

if we choose ϵ_0 sufficiently small.

(3). By (3.14), Lemma 4.1 Property $(a)(b)$ we have

$$\begin{aligned}
E(t) &\le \int_0^\tau \max_L(|A||H|+|H|^2)\,ds + \int_\tau^t \max_L(|A||H|+|H|^2)\,ds \\
&\le 4\Lambda_0\epsilon_0\tau + 4\epsilon_0^2\tau + 3\Lambda_0\epsilon_0^{\frac{1}{n+2}}\frac{2n(n+2)}{\bar{R}} + \epsilon_0^{\frac{2}{n+2}}\frac{n(n+2)}{\bar{R}} \\
&\le \frac{1}{n+1}\log\frac{3}{2}, \quad t\in[0, T],
\end{aligned}$$

where ϵ_0 is small enough. Thus, by Lemma 3.4 L_t is $\frac{2}{3}\kappa_0$-noncollapsed on the scale r_0 for $t\in[0, T]$.

□

Now we can prove the following stability result, which needs the noncollapsing condition of the initial submanifold. This condition can be removed by the comparison theorem in Theorem 1.1.

Theorem 4.3. *Let $(M,\bar g)$ be a complete Kähler-Einstein manifold satisfying (1.1) with scalar curvature $\bar R < 0$, and L be a compact Lagrangian submanifold smoothly immersed in M. For any $\kappa_0, r_0, V_0, \Lambda_0 > 0$, there exists $\epsilon_0 = \epsilon_0(\kappa_0, r_0, V_0, \Lambda_0, \bar R, K_5)$ such that if L is κ_0-noncollapsed on the scale r_0 and satisfies*

$$\mathrm{Vol}(L) \le V_0, \quad |A| \le \Lambda_0, \quad |H| \le \epsilon_0,$$

then the Lagrangian mean curvature flow with the initial data L will converge exponentially fast to a minimal Lagrangian submanifold in M.

Proof. . Suppose that $L_0 \in \mathcal{A}(\kappa_0, r_0, \Lambda_0, \epsilon_0)$ for any positive constants κ_0, r_0, Λ_0 and small ϵ_0 which will be chosen later. Define

$$t_0 = \sup\Big\{ t>0 \,\Big|\, L_s \in \mathcal{A}(\frac{1}{3}\kappa_0, r_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}}), \quad s\in[0,t) \Big\}.$$

Suppose that $t_0 < +\infty$. By Lemma 4.2, there exists $\epsilon_0 = \epsilon_0(\kappa_0, r_0, \Lambda_0, n, K_5, V_0)$ such that $L_t \in \mathcal{A}(\frac{2}{3}\kappa_0, r_0, 3\Lambda_0, \epsilon_0^{\frac{1}{n+2}})$ for all $t\in[0,t_0)$. Moreover, by Lemma 4.2 again the solution L_t can be extended to $[0, t_0+\delta]$ such that $L_t \in \mathcal{A}(\frac{1}{3}\kappa_0, r_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}})$, which contradicts the definition of t_0. Thus, $t_0 = +\infty$ and

$$L_t \in \mathcal{A}(\frac{1}{3}\kappa_0, r_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}}), \quad t\in[0,\infty).$$

By Lemma 4.2 the mean curvature vector will decay exponentially to zero and the flow will converge to a smooth minimal Lagrangian submanifold. The theorem is proved.

□

We can finish the proof of Theorem 1.1 as follows:

Proof of Theorem 1.1. We will show that under the assumption of Theorem 1.1, the flow L_t will satisfies all the conditions in Theorem 4.3 after a short time. Suppose that the initial Lagrangian submanifold L satisfies (1.2), by Lemma 3.6 there exists $T = T(n, \Lambda, K_1)$ such that

$$|A|(t) \le 2\Lambda, \quad t\in[0,T]. \tag{4.9}$$

We claim that there exists $t_0 = t_0(n,\Lambda,K_1) < T$ such that the L^2 norm of the mean curvature vector satisfies

$$\int_{L_t} |H|^2 \, d\mu_t \le 2\epsilon_0, \quad t\in[0,t_0]. \tag{4.10}$$

In fact, by (3.7) in Lemma 3.1 we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{L_t} |H|^2 d\mu_t &\le \int_{L_t} 2\bar R_{ij}H^iH^j - 2H^iH^jH^kh^i_{jk} - |H|^4 \\ &\le \Big(\frac{\bar R}{n} + 4\Lambda^2\Big)\int_{L_t} |H|^2 d\mu_t, \end{aligned} \tag{4.11}$$

where we used (4.9) and the inequality

$$2H^iH^jH^kh^i_{jk} \leq 4\Lambda^2|H|^2 + |H|^4.$$

Thus, we have

$$\int_{L_t} |H|^2 d\mu_t \leq e^{4\Lambda^2 t} \int_{L_0} |H|^2 d\mu_0 \leq \epsilon_0 e^{4\Lambda^2 t}, \quad t \in [0, T]. \tag{4.12}$$

which proves (4.10) if we choose t_0 sufficiently small.

Now we prove that there exist $\kappa_0, r_0 > 0$ such that L_t is κ_0-noncollapsed on the scale r_0 for $t \in [\frac{1}{2}t_0, t_0]$. In fact, by Proposition 2.2 in [2] or Theorem 2.1 in [7] the injectivity radius of L is bounded from below

$$inj(L_t) \geq \iota, \quad t \in [\frac{1}{2}t_0, t_0] \tag{4.13}$$

for some constant $\iota = \iota(n, \Lambda, K_0, \iota_0)$. By (4.9) and by Gauss equation the intrinsic curvature of L_t is uniformly bounded

$$|Rm| \leq C(K_0, \Lambda), \quad t \in [\frac{1}{2}t_0, t_0]. \tag{4.14}$$

By (4.13)(4.14) together with the volume comparison theorem, there exist $\kappa_0 = \kappa_0(n, \iota_0, K_0, \Lambda)$ and $r_0 = r_0(n, \iota_0, K_0, \Lambda)$ such that L_t is κ_0-noncollapsed on the scale r_0 for all $t \in [\frac{1}{2}t_0, t_0]$.

By (4.9) and Lemma 3.7 the derivative of the second fundamental form is uniformly bounded

$$|\nabla A| \leq C_1(n, \Lambda, K_2), \quad t \in [\frac{1}{2}t_0, t_0].$$

Now we can apply Lemma 3.5 to show that

$$|H|(t) \leq \Big(\frac{1}{\sqrt{\kappa_0}} + 2C_1\Big)(2\epsilon_0)^{\frac{1}{n+2}} \quad t \in [\frac{1}{2}t_0, t_0].$$

In summary, all the conditions in Theorem 4.3 are satisfied for $L_t(t \in [\frac{1}{2}t_0, t_0])$, and thus Theorem 1.1 is proved.

□

5 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. The idea of the proof is similar to that of Theorem 1.1 but more involved since we need to consider the evolution of the first eigenvalue.

For any positive constants $\delta, \kappa, r, \Lambda, \epsilon$, we define the following subspace of Lagrangian submanifolds in M by

$$\mathcal{B}(\kappa, r, \delta, \Lambda, \epsilon) = \Big\{ L \;\Big|\; L \text{ is } \kappa\text{-noncollapsed on the scale } r \text{ with } \lambda_1 \geq \tfrac{\bar{R}}{2n} + \delta, \;\; |A|(t) \leq \Lambda, \;\; |H|(t) \leq \epsilon \Big\}.$$

Lemma 5.1. *If the initial Lagrangian submanifold $L_0 \in \mathcal{B}(\kappa, r, \delta, \Lambda, \epsilon)$, then there exists $\tau = \tau(n, \Lambda, \delta, K_2, \bar{R}) > 0$ such that $L_t \in \mathcal{B}(\frac{1}{2}\kappa, r, \frac{2\delta}{3}, 2\Lambda, 2\epsilon)$ for $t \in [0, \tau]$.*

Proof. This result follows directly from Lemma 3.4 and Lemma 3.6. $\square$

Lemma 5.2. *For any $\kappa_0, r_0, \delta_0, \Lambda_0, V_0, T > 0$ there exists $\epsilon_0 = \epsilon_0(\kappa_0, r_0, \delta_0, \bar{R}, \Lambda_0, n, K_5, V_0) > 0$ such that if the solution $L_t (t \in [0, T])$ of the Lagrangian mean curvature flow satisfies*

1. $L_0 \in \mathcal{B}(\kappa_0, r_0, \delta_0, \Lambda_0, \epsilon_0)$ *and* $\mathrm{Vol}(L_0) \leq V_0$,
2. $L_t \in \mathcal{B}(\frac{1}{3}\kappa_0, r_0, \frac{1}{3}\delta_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}})(t \in [0, T])$,

Then we have the following properties

(a) The mean curvature vector satisfies

$$\max_{L_t} |H|(t) \leq \epsilon_0^{\frac{1}{n+2}} e^{-\frac{\delta_0}{2(n+2)}t}, \quad t \in [\tau, T].$$

(b) The second fundamental form

$$\max_{L_t} |A| \leq 3\Lambda_0, \quad t \in [0, T].$$

(c) L_t is $\frac{2}{3}\kappa_0$-noncollapsed on the scale r_0 for $t \in [0, T]$.

(d) The first eigenvalue

$$\lambda_1(t) \geq \frac{\bar{R}}{2n} + \frac{\delta_0}{2}, \quad t \in [0, T].$$

Thus, the solution $L_t \in \mathcal{B}(\frac{2}{3}\kappa_0, r_0, \frac{\delta_0}{2}, 3\Lambda_0, \epsilon_0^{\frac{1}{n+2}})$ for $t \in [0, T]$, and by Lemma 5.1 we can extend the solution to $[0, T + \delta]$ such that $L_t \in \mathcal{B}(\frac{1}{3}\kappa_0, r_0, \frac{1}{3}\delta_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}})(t \in [0, T + \delta])$ for some $\delta = \delta(n, \Lambda_0, K_1) > 0$.

Proof. (a). By assumption 2, the first eigenvalue satisfies $\lambda_1(t) \geq \frac{\bar{R}}{2n} + \frac{\delta_0}{3}$. Thus, we have

$$\lambda_1 - \frac{\bar{R}}{2n} - 12\Lambda_0 \epsilon^{\frac{1}{n+2}} \geq \frac{\delta_0}{4}, \quad t \in [0, T]$$

when ϵ_0 is small enough. Thus, by (3.2) in Lemma 3.1 the mean curvature vector satisfies

$$\int_{L_t} |H|^2 d\mu_t \leq e^{-\frac{\delta_0}{2}t} \int_{L_0} |H|^2 d\mu_0 \leq V_0 \epsilon_0^2 e^{-\frac{\delta_0}{2}t}, \quad t \in [0, T]. \tag{5.1}$$

Note that $L_t \in \mathcal{B}(\frac{1}{3}\kappa_0, r_0, \frac{1}{3}\delta_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}})$ for $t \in [0, T]$, by Lemma 3.7 there is a constant $C_1 = C_1(n, \Lambda_0, K_2)$ such that

$$|\nabla A|(t) \leq C_1(n, \Lambda_0, K_2, \tau), \quad t \in [\tau, T]. \tag{5.2}$$

Here we can choose $\tau=\tau(n,\Lambda_0,\delta_0,K_2,\bar{R})$ in Lemma 5.1. Thus, by Lemma 3.5 and (5.1)(5.2) we have

$$|H|(t)\leq\Big(\sqrt{\frac{3}{\kappa_0}}+C_1\Big)V_0^{\frac{1}{n+2}}\epsilon_0^{\frac{2}{n+2}}e^{-\frac{\delta_0}{2(n+2)}t},\quad t\in[\tau,T]. \tag{5.3}$$

where we have used the fact that L_t is $\frac{\kappa}{3}$-noncollapsed on the scale r_0 and $V_0\epsilon_0^2\leq r_0^{n+2}$ if ϵ_0 is small enough. Thus, if ϵ_0 is small such that $\Big(\sqrt{\frac{3}{\kappa_0}}+C_1\Big)V_0^{\frac{1}{n+2}}\epsilon_0^{\frac{1}{n+2}}\leq 1$, then we have

$$|H|(t)\leq\epsilon_0^{\frac{1}{n+2}}e^{-\frac{\delta_0}{2(n+2)}t},\quad t\in[\tau,T].$$

(b). By Lemma 3.7 there exist some constants $C_k=C_k(n,\Lambda_0,K_{k+1})$ such that

$$|\nabla^kA|(t)\leq C_k(n,\Lambda_0,K_{k+1},\tau),\quad t\in[\tau,T]. \tag{5.4}$$

By Lemma 3.7 and Property (a), we have

$$\int_{L_t}|\nabla^2H|^2d\mu_t\leq\int_{L_t}|H||\nabla^4H|d\mu_t\leq V_0C_4\epsilon_0^{\frac{1}{n+2}}e^{-\frac{\delta_0}{2(n+2)}t},\quad t\in[\tau,T].$$

Thus, by Lemma 3.5 we have

$$|\nabla^2H|\leq\Big(\sqrt{\frac{3}{\kappa_0}}+C_3\Big)C_4^{\frac{1}{n+2}}V_0^{\frac{1}{n+2}}\epsilon_0^{\frac{1}{(n+2)^2}}e^{-\frac{\delta_0}{2(n+2)^2}t},\quad t\in[\tau,T]. \tag{5.5}$$

Recall that by Lemma 2.3 $|A|$ satisfies the inequality

$$\frac{\partial}{\partial t}|A|\leq|\nabla^2H|+c(n)|A|^2|H|+|\bar{R}m||H|. \tag{5.6}$$

Thus, by Lemma 5.1, (5.5)(5.6) and (a) we have

$$\begin{aligned}|A|(t)&\leq|A|(\tau)+\int_\tau^t|\nabla^2H|+(K_0+|A|^2)|H|\\&\leq 2\Lambda_0+\Big(\sqrt{\frac{3}{\kappa_0}}+C_3\Big)C_4^{\frac{1}{n+2}}V_0^{\frac{1}{n+2}}\epsilon_0^{\frac{1}{(n+2)^2}}\frac{2(n+2)^2}{\delta_0}\\&\quad+(K_0+36\Lambda_0^2)\epsilon_0^{\frac{1}{n+2}}\frac{2(n+2)}{\delta_0}\\&\leq 3\Lambda_0,\end{aligned} \tag{5.7}$$

if we choose ϵ sufficiently small.

(c). By (3.14), Lemma 5.1 Property $(a)(b)$ we have

$$\begin{aligned}E(t)&\leq\int_0^\tau\max_L(|A||H|+|H|^2)\,ds+\int_\tau^t\max_L(|A||H|+|H|^2)\,ds\\&\leq 4\Lambda_0\epsilon_0\tau+4\epsilon_0^2\tau+3\Lambda_0\epsilon_0^{\frac{1}{n+2}}\frac{2(n+2)}{\delta_0}+\epsilon_0^{\frac{2}{n+2}}\frac{(n+2)}{\delta_0}\\&\leq\frac{1}{n+1}\log\frac{3}{2},\quad t\in[0,T],\end{aligned}$$

where ϵ_0 is small enough. Thus, by Lemma 3.4 L_t is $\frac{2}{3}\kappa_0$-noncollapsed on the scale r_0 for $t \in [0, T]$.

(d). By (5.4) and Property (a) we have

$$\int_{L_t} |\nabla H|^2 \le \int_{L_t} |H||\nabla^2 H| \le V_0 C_2 \epsilon_0^{\frac{1}{n+2}} e^{-\frac{\delta_0}{2(n+2)}t}, \quad t \in [\tau, T]$$

Thus, by Lemma 3.5 we have

$$\begin{aligned} |\nabla H| &\le \Big(\sqrt{\frac{3}{\kappa_0}} + C_2\Big) C_2^{\frac{1}{n+2}} V_0^{\frac{1}{n+2}} \epsilon_0^{\frac{1}{(n+2)^2}} e^{-\frac{\delta_0}{2(n+2)^2}t} \\ &\le \epsilon_0^{\frac{1}{2(n+2)^2}} e^{-\frac{\delta_0}{2(n+2)^2}t}, \quad t \in [\tau, T], \end{aligned} \tag{5.8}$$

if ϵ_0 is sufficiently small. Thus, we have

$$|H| + |\nabla H| \le 2\epsilon_0^{\frac{1}{2(n+2)^2}} e^{-\frac{\delta_0}{2(n+2)^2}t} \tag{5.9}$$

Note that by Lemma 5.1 the first eigenvalue

$$\lambda_1(t) \ge \frac{\bar{R}}{2n} + \frac{2\delta_0}{3}, \quad t \in [0, \tau].$$

Thus, by (3.10) in Lemma 3.2 we have

$$\begin{aligned} \sqrt{\lambda_1(t)} &\ge \sqrt{\lambda_1(\tau)} e^{-\frac{(n+2)^2}{\delta_0}\cdot\left(24\Lambda_0 \epsilon_0^{\frac{1}{2(n+2)^2}} + 4\epsilon_0^{\frac{1}{(n+2)^2}}\right)} - \frac{2(n+2)^2}{\delta_0}(K_0 + 6\Lambda_0)\cdot 2\epsilon_0^{\frac{1}{2(n+2)^2}} \\ &\ge \sqrt{\frac{\bar{R}}{2n} + \frac{2\delta_0}{3}} e^{-\frac{(n+2)^2}{\delta_0}\cdot\left(24\Lambda_0 \epsilon_0^{\frac{1}{2(n+2)^2}} + 4\epsilon_0^{\frac{1}{(n+2)^2}}\right)} - \frac{2(n+2)^2}{\delta_0}(K_0 + 6\Lambda_0)\cdot 2\epsilon_0^{\frac{1}{2(n+2)^2}}. \end{aligned}$$

Thus, if ϵ_0 is small enough we have

$$\lambda_1(t) \ge \frac{\bar{R}}{2n} + \frac{\delta_0}{2}, \quad t \in [0, T].$$

The lemma is proved.

□

As in the proof of Theorem 1.1, we can see that Theorem 1.2 follows directly from Lemma 3.2 and the result:

Theorem 5.3. *Let $(M, \bar{g})$ be a complete Kähler-Einstein manifold satisfying (1.1) with scalar curvature $\bar{R} \ge 0$, and L be a compact Lagrangian submanifold smoothly immersed in M. For any $\kappa_0, r_0, V_0, \Lambda_0, \delta_0 > 0$, there exists $\epsilon_0 = \epsilon_0(\kappa_0, r_0, V_0, \Lambda_0, \bar{R}, \delta_0, K_5) > 0$ such that if*

1. *the mean curvature form of L is exact,*

2. *L is κ_0-noncollapsed on the scale r_0,*

3. L satisfies
$$\lambda_1 \geq \frac{\bar{R}}{2n} + \delta_0, \quad \mathrm{Vol}(V) \leq V_0, \quad |A| \leq \Lambda_0, \quad |H| \leq \epsilon_0,$$
where λ_1 is the first eigenvalue of the Laplacian operator with respect to the induced metric on L,

then the Lagrangian mean curvature flow with the initial data L will converge exponentially fast to a minimal Lagrangian submanifold in M.

Proof. . Suppose that $L_0 \in \mathcal{B}(\kappa_0, r_0, \delta_0, \Lambda_0, \epsilon_0)$ for any positive constants $\kappa_0, r_0, \delta_0, \Lambda_0$ and small ϵ_0 which will be chosen later. Define

$$t_0 = \sup\Big\{t > 0 \;\Big|\; L_s \in \mathcal{B}(\frac{1}{3}\kappa_0, r_0, \frac{1}{3}\delta_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}}), \quad s \in [0, t)\Big\}.$$

Suppose that $t_0 < +\infty$. By Lemma 5.2, there exists $\epsilon_0 = \epsilon_0(\kappa_0, r_0, \delta_0, \bar{R}, \Lambda_0, n, K_5, V_0) > 0$ such that $L_t \in \mathcal{B}(\frac{2}{3}\kappa_0, r_0, \frac{\delta_0}{2}, 3\Lambda_0, \epsilon_0^{\frac{1}{n+2}})$ for all $t \in [0, t_0)$. Moreover, by Lemma 5.2 again the solution L_t can be extended to $[0, t_0 + \delta]$ such that $L_t \in \mathcal{B}(\frac{1}{3}\kappa_0, r_0, \frac{1}{3}\delta_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}})$, which contradicts the definition of t_0. Thus, $t_0 = +\infty$ and

$$L_t \in \mathcal{B}(\frac{1}{3}\kappa_0, r_0, \frac{1}{3}\delta_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}}), \quad t \in [0, \infty).$$

By Lemma 5.2 the mean curvature vector will decay exponentially to zero and the flow will converge to a smooth minimal Lagrangian submanifold. The theorem is proved.

□

6 Proof of Theorem 1.4

In this section, we will introduce some definitions related to the deformation of a Lagrangian submanifold, and prove the exponential decay of the mean curvature vector under the Lagrangian mean curvature flow with some special initial data. The idea of the argument is very similar to Kähler-Ricci flow in a Kähler-Einstein manifold with nonzero holomorphic vector fields(cf. [3][4]).

6.1 Deformation of minimal Lagrangian submanifolds

Let $(M, \bar{g})$ be a Kähler-Einstein manifold. First we give some definitions(cf. [14]):

Definition 6.1. *(1). Let $L \subset M$ be a Lagrangian submanifold and X be a vector field along L. X is called a Lagrangian(resp. hamiltonian) variation if its associated one form*

$$\alpha_X = i_X \bar{\omega}$$

is closed(resp. exact), where $\bar{\omega}$ is the Kähler form of the metric $\bar{g}$ on M.

(2). A smooth family ϕ_s of immersions of L into M is called a Lagrangian (resp. hamiltonian) deformation if its derivative

$$X = \frac{\partial \phi_s(L)}{\partial s}$$

is Lagrangian (resp. hamiltonian) for each s.

In the following, we assume that $\phi_0 : L \to M$ is a smooth minimal Lagrangian submanifold into a Kähler-Einstein manifold $(M, \bar{g})$, and $X = J\nabla f_0$ is a hamiltonian variation of $L_0 = \phi_0(L)$. We remind that the notation L_0 has different meaning in previous sections, and the readers should not confuse it.

We can extend the vector X to a neighborhood of L_0 in M such that it is still hamiltonian. Let $\phi_s : L \to M(s \in (-\eta, \eta))$ be a family of hamiltonian deformations generated by X and we write $L_s = \phi_s(L_0)$. For the hamiltonian deformation L_s, we have the following result:

Lemma 6.2. *Let f_s be a smooth function such that*

$$\frac{\partial L_s}{\partial s} = J\nabla f_s, \tag{6.1}$$

then the Lagrangian angle θ_s of L_s satisfies

$$\frac{\partial \theta_s}{\partial s} = -\Delta_s f_s - \frac{\bar{R}}{2n} f_s. \tag{6.2}$$

Proof. Let $\{e_1, \cdots, e_n\}$ be a normal coordinate frame on L_s with $e_i = \partial_i \phi_s$. Since L_s is Lagrangian for each s, the vectors $Je_1, \cdots, Je_n$ are orthogonal to L. The induced metric on L_s is $g_{ij} = \bar{g}(e_i, e_j)$. By (6.1) we have

$$\frac{\partial g_{ij}}{\partial s} = 2\nabla^k f_s h^k_{ij}. \tag{6.3}$$

By the same calculation as in Lemma 2.3, the second fundamental form satisfies

$$\frac{\partial}{\partial t} h^k_{ij} = -\nabla_i \nabla_j \nabla^k f_s + \nabla^m f_s h^m_{jl} h^k_{il} + \nabla^l f_s h^l_{mk} h^m_{ij} - \nabla^m f_s \bar{R}_{\underline{k}j\underline{m}i}$$

and the mean curvature vector

$$\frac{\partial}{\partial t} H^i = -\Delta \nabla^i f_s - \nabla^j f_s h^j_{lm} h^i_{ml} - \nabla^j f_s \bar{R}_{\underline{i}l\underline{j}l} + \nabla^j f_s H^k h^i_{kj}. \tag{6.4}$$

Since L_s is a hamiltonian deformation, we can write $H^i = \nabla^i \theta_s$. Thus, by the same calculation in the proof of Lemma 2.3 and (6.4) we have

$$\frac{\partial \theta_s}{\partial s} = -\Delta_s f_s - \frac{\bar{R}}{2n} f_s.$$

The lemma is proved. $\square$

Recall that a minimal Lagrangian submanifold is called hamiltonian stable (resp. strictly stable), if for any hamiltonian variation X the second variation along X of the volume functional is nonnegative (resp. positive). When M is a Kähler-Einstein manifold with scalar curvature $\bar{R}$, Oh [14] proved that a compact minimal Lagrangian submanifold L is hamiltonian stable if and only if the first eigenvalue of the Laplacian operator on L has $\lambda_1 \geq \frac{\bar{R}}{2n}$.

Now we introduce the definition:

Definition 6.3. *A nonzero vector field X is called an essential hamiltonian variation of a Lagrangian submanifold L_0 , if X can be written as $X = J\nabla f$ where $f \notin E_{\lambda_1}$, where E_{λ_1} is the first eigenspace of the Laplacian operator Δ on L.*

For an essential hamiltonian vector X on a minimal Lagrangian submanifold L_0, we can show that L_0 is strictly hamiltonian stable along the variation X in the following sense:

Lemma 6.4. *Let $\phi_s : L \to M$ be a hamiltonian deformation of a minimal Lagrangian submanifold $L_0 = \phi_0(L)$ with $\lambda_1 = \frac{\bar{R}}{2n}$ and $X = \frac{\partial \varphi}{\partial s}|_{s=0}$. Then X is an essential hamiltonian variation on L_0 if and only if*

$$\frac{d^2}{ds^2}\mathrm{Vol}(L_s)\Big|_{s=0} > 0.$$

Proof. Let $X_s = \frac{\partial \phi_s}{\partial s}$. Since L_s is a hamiltonian deformation, we can find smooth functions f_s such that $X_s = \nabla f_s$. Now we calculate

$$\begin{aligned} \frac{d}{ds}\mathrm{Vol}(L_s) &= \int_{L_s} \frac{1}{2} g^{ij} \frac{\partial g_{ij}}{\partial s}\, d\mu_s \\ &= \int_{L_s} -\theta_s \Delta_s f_s \, d\mu_s, \end{aligned}$$

where we used (6.3). Thus, the second variation of the volume is

$$\frac{d^2}{ds^2}\mathrm{Vol}(L_s)\Big|_{s=0} = \int_{L_0} \Big(\Delta_0 f_0 + \frac{\bar{R}}{2n} f_0\Big) \Delta_0 f_0 \, d\mu_0 \tag{6.5}$$

where we used the equality (6.2) and the fact that L_0 is minimal. By the eigenvalue decomposition, we can assume that

$$f_0 = \sum_{i=1}^{\infty} a_i \eta_i,$$

where the functions η_i satisfies

$$-\Delta_0 \eta_i = \lambda_i \eta_i, \quad \int_{L_0} \eta_i^2 \, d\mu_0 = 1$$

for the eigenvalues $\lambda_1 < \lambda_2 < \cdots$. Thus, (6.5) can be written as

$$\frac{d^2}{ds^2}\mathrm{Vol}(L_s)\Big|_{s=0} = \sum_{i=1}^{\infty} a_i^2 \lambda_i (\lambda_i - \frac{\bar{R}}{2n}) \geq 0, \tag{6.6}$$

where the equality holds if and only if $a_i = 0$ for all $i \geq 2$, which says $f_0 \in E_{\lambda_1}$. The lemma is proved. $\square$

6.2 Exponential decay of the mean curvature vector

To proceed further, we need the following compactness result for mean curvature flow.

Proposition 6.5. *(cf. [2]) Let $\phi_k(t) : L \subset M$ be a sequence of mean curvature flow from a compact submanifold L to a compact Riemannian manifold M with uniformly bounded second fundamental forms*

$$|A_k|(t) \leq C, \quad \forall\, t \in [0, T].$$

Then there exists a sequence of $\phi_k(t)$ which converges to a mean curvature flow $\phi_\infty(t)(t \in (0, T)))$ and $L_\infty = \phi_\infty(t)(L)$ is a smooth Riemannian manifold.

Proof. The proposition is proved by Chen-He in [2] for the case when the ambient manifold is the Euclidean space. For a general compact ambient manifold M, we can embed M isometrically into $\mathbb{R}^N$ for some large N and the corresponding mean curvature of the submanifold $\phi_k(L)(t)$ in $\mathbb{R}^N$ is still uniformly bounded. Thus, we can apply Chen-He's theorem and the proposition is proved.

□

We denote by $L_{s,t} = \phi_{s,t}(L_0)(t \in [0, T])$ the Lagrangian mean curvature flow with the initial data L_s. Since L_s is a hamiltonian deformation of L_0, the mean curvature form of L_s is exact for each s. Thus, the mean curvature form of $L_{s,t}$ is also exact, and we denote the Lagrangian angle by $\theta_{s,t}$. Suppose that the deformation L_s is sufficiently close to L_0 in the following sense

$$\|\phi_s - \phi_0\|_{C^3} \leq \epsilon_0 \tag{6.7}$$

for small ϵ_0 which will be determined later. The next lemma shows that $\theta_{s,t}$ satisfies certain inequality if L_s is sufficiently close to L_0 :

Lemma 6.6. *Let $X = J\nabla f_0$ be an essential hamiltonian variation of L_0, where L_0 is a minimal Lagrangian submanifold with the first eigenvalue $\lambda_1 = \frac{\bar{R}}{2n}$. For any $\Lambda > 0$, there exists $\epsilon_0 = \epsilon_0(L_0, X, M) > 0$ and $\delta_0 > 0$ such that if $L_{s,t}$ satisfies*

$$|A_s|(t) \leq \Lambda, \quad |H_s|(t) \leq \epsilon_0, \quad \forall\, t \in [0, T] \tag{6.8}$$

then the Lagrangian angle $\theta_{s,t}$ of $L_{s,t}$ satisfies

$$\int_{L_{s,t}} |\Delta\theta_{s,t}|^2 \geq (\frac{\bar{R}}{2n} + \delta_0) \int_{L_{s,t}} |\nabla\theta_{s,t}|^2, \quad t \in [0, T]. \tag{6.9}$$

Thus, we have

$$\frac{\partial}{\partial t} \int_{L_{s,t}} |H_{s,t}|^2 \leq -2(\delta_0 - \Lambda\epsilon_0) \int_{L_{s,t}} |H_{s,t}|^2, \quad t \in [0, T]. \tag{6.10}$$

Proof. Suppose that (6.9) doesn't hold, there exist some constants $s_i \to 0, \delta_i \to 0$ and $t_i \in [0, T]$ such that

$$|A_{s_i}|(t) \leq \Lambda, \quad |H_{s_i}|(t) \to 0, \quad t \in [0, T] \tag{6.11}$$

and

$$\int_{L_{s_i,t_i}} |\Delta\theta_{s_i,t_i}|^2 \le (\frac{\bar{R}}{2n}+\delta_i)\int_{L_{s_i,t_i}} |\nabla\theta_{s_i,t_i}|^2. \tag{6.12}$$

By (6.11) and Proposition 6.5 a sequence of the Lagrangian mean curvature flow $L_{s_i,t}(t \in (0,T))$ will converge to a limit Lagrangian mean curvature flow $L_{\infty,t}$ smoothly for $t \in (0,T)$. Since the initial submanifolds L_{s_i} satisfies (6.7), the limit flow has $L_\infty(t) \to L_0$ in $C^{2,\alpha}$ as t goes to zero. By (6.11)again the mean curvature of $L_\infty(t)(t \in (0,T))$ are identically zero and by the uniqueness of mean curvature flow we have

$$L_{s_i,t} \to L_{\infty,t} = L_0, \quad t \in [0,T].$$

Note that by (6.12) we have

$$\int_{L_{s_i,t_i}} |\Delta\frac{1}{s_i}\theta_{s_i,t_i}|^2 \le (\frac{\bar{R}}{2n}+\delta_i)\int_{L_{s_i,t_i}} |\nabla\frac{1}{s_i}\theta_{s_i,t_i}|^2.$$

Since L_0 is minimal, we can take $s_i \to 0$ to get

$$\int_{L_0} \Big|\Delta\frac{\partial\theta_{s,t}}{\partial s}\Big|_{(0,t_i)}\Big|^2 \le \frac{\bar{R}}{2n}\int_{L_0} \Big|\nabla\frac{\partial\theta_{s,t}}{\partial s}\Big|_{(0,t_i)}\Big|^2. \tag{6.13}$$

On the other hand, by Lemma 2.4 the Lagrangian angle $\theta_{s,t}$ satisfies

$$\frac{\partial\theta_{s,t}}{\partial t} = \Delta_{s,t}\theta_{s,t} + \frac{\bar{R}}{2n}\theta_{s,t}.$$

Since L_0 is minimal, we can take $s = s_i \to 0$ to derive

$$\frac{\partial}{\partial t}\frac{\partial\theta_{s,t}}{\partial s}\Big|_{s=0} = \Delta_0\frac{\partial\theta_{s,t}}{\partial s}\Big|_{s=0} + \frac{\bar{R}}{2n}\frac{\partial\theta_{s,t}}{\partial s}\Big|_{s=0}. \tag{6.14}$$

By the eigenvalue decomposition as in Lemma 6.4, we have

$$\frac{\partial\theta_{s,t}}{\partial s}\Big|_{(s,t)=(0,0)} = -\Delta_0 f_0 - \frac{\bar{R}}{2n}f_0 \perp E_{\lambda_1}.$$

Now we claim that

$$\frac{\partial\theta_{s,t}}{\partial s}\Big|_{s=0} \perp E_{\lambda_1}, \quad t \in [0,T]. \tag{6.15}$$

In fact, for any function $\eta \in E_{\lambda_1}$ by (6.14) we have

$$\begin{aligned} \frac{\partial}{\partial t}\int_{L_0} \eta\frac{\partial\theta_{s,t}}{\partial s}\Big|_{s=0} &= \int_{L_0} \eta\Big(\Delta_0\frac{\partial\theta_{s,t}}{\partial s}\Big|_{s=0} + \frac{\bar{R}}{2n}\frac{\partial\theta_{s,t}}{\partial s}\Big|_{s=0}\Big) \\ &= \int_{L_0} \Big(\Delta_0\eta + \frac{\bar{R}}{2n}\eta\Big)\frac{\partial\theta_{s,t}}{\partial s}\Big|_{s=0} \\ &= 0, \end{aligned}$$

which proves (6.15).

Note that

$$\frac{\partial\theta_{s,t}}{\partial s}\Big|_{(s,t)=(0,0)}=-\Delta_0 f_0-\frac{\bar{R}}{2n}f_0\neq 0,$$

since X is an essential hamiltonian variation. Thus, by (6.14) we can see that $\frac{\partial\theta_{s,t}}{\partial s}\Big|_{s=0}$ is nonzero for all $t\in[0,T]$, and (6.15) implies that

$$\int_{L_0}\Big|\Delta\frac{\partial\theta_{s,t}}{\partial s}\Big|_{s=0}\Big|^2\geq\lambda_2\int_{L_0}\Big|\nabla\frac{\partial\theta_{s,t}}{\partial s}\Big|_{s=0}\Big|^2, \tag{6.16}$$

where the second eigenvalue $\lambda_2>\lambda_1=\frac{\bar{R}}{2n}$. Note that (6.16) contradicts (6.13), and (6.9) is proved.

Recall that by (3.7) in Lemma 3.1 we have

$$\begin{aligned}\frac{\partial}{\partial t}\int_{L_{s,t}}|H|^2 &\leq \int_{L_{s,t}}-2|\nabla_i H^i|^2+\frac{\bar{R}}{n}|H|^2+2\Lambda\epsilon_0|H|^2\\ &= \int_{L_{s,t}}-2|\Delta\theta_{s,t}|^2+\Big(\frac{\bar{R}}{n}+2\Lambda\epsilon_0\Big)|\nabla\theta_{s,t}|^2\\ &\leq -2(\delta_0-\Lambda\epsilon_0)\int_{L_{s,t}}|H|^2. \end{aligned}\tag{6.17}$$

Thus, (6.10) is proved. □

6.3 Proof of Theorem 1.4

In this section, we will prove Theorem 1.4 by using the same argument as in the proof of Theorem 1.1 and 1.2. Since L_0 is a smooth minimal Lagrangian submanifold, we can find $\kappa_0, r_0>0$ such that L_0 is $2\kappa_0$-noncollapsed on the scale r_0. Thus, by the assumption of Theorem 1.4 we can choose ϵ_0 small enough such that L_s is κ_0-noncollapsed on the scale r_0 and $L_s\in\mathcal{A}(\kappa_0,r_0,\Lambda_0,\epsilon_0)$ for some constant $\Lambda_0>0$, where $\mathcal{A}(\kappa,r,\Lambda,\epsilon)$ is the following subspace of Lagrangian submanifolds in M defined by

$$\mathcal{A}(\kappa,r,\Lambda,\epsilon)=\Big\{L\;\Big|\;L\text{ is }\kappa\text{-noncollapsed on the scale }r\text{ with }|A|(t)\leq\Lambda,\ |H|(t)\leq\epsilon\Big\}.$$

Consider the solution $L_{s,t}$ of the Lagrangian mean curvature flow with the initial data L_s, we have

Lemma 6.7. *If the initial Lagrangian submanifold $L_s\in\mathcal{A}(\kappa,r,\Lambda,\epsilon)$, then there exists $\tau=\tau(n,\Lambda,K_1)$ such that $L_{s,t}\in\mathcal{A}(\frac{1}{2}\kappa,r,2\Lambda,2\epsilon)$ for $t\in[0,\tau]$.*

Proof. This result follows directly from Lemma 3.4 and Lemma 3.6. □

Lemma 6.8. *For any $\kappa_0,r_0,\Lambda_0,V_0,T>0$ there exists $\epsilon_0=\epsilon_0(\kappa_0,r_0,\Lambda_0,n,K_5,V_0)$ such that if the solution $L_{s,t}(t\in[0,T])$ of the Lagrangian mean curvature flow satisfies*

1. *$L_s\in\mathcal{A}(\kappa_0,r_0,\Lambda_0,\epsilon_0)$ and $\mathrm{Vol}(L_s)\leq V_0$,*

2. $L_{s,t}\in\mathcal{A}(\frac{1}{3}\kappa_0, r_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}})(t\in[0,T])$*,*

Then we have the following properties

(a) The mean curvature vector satisfies

$$\max_{L_{s,t}}|H_{s,t}|\le\epsilon_0^{\frac{1}{n+2}}e^{-\frac{\delta_0}{n+2}t},\quad t\in[\tau,T].$$

(b) The second fundamental form

$$\max_{L_{s,t}}|A_{s,t}|\le 3\Lambda_0,\quad t\in[0,T].$$

(c) $L_{s,t}$ *is* $\frac{2}{3}\kappa_0$*-noncollapsed on the scale* r_0 *for* $t\in[0,T]$*.*

Thus, the solution $L_{s,t}\in\mathcal{A}(\frac{2}{3}\kappa_0, r_0, 3\Lambda_0, \epsilon_0^{\frac{1}{n+2}})$ *for* $t\in[0,T]$*, and by Lemma 6.7 we can extend the solution to* $[0,T+\delta]$ *such that* $L_{s,t}\in\mathcal{A}(\frac{1}{3}\kappa_0, r_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}})(t\in[0,T+\delta])$ *for some* $\delta=\delta(n,\Lambda_0,K_1)>0$*.*

Proof. (a). For any $\Lambda_0>0$, by Lemma 6.6 we can choose $\epsilon_0=\epsilon_0(X,\Lambda,L_0,M)$ small enough such that the mean curvature vector satisfies

$$\int_{L_{s,t}}|H_{s,t}|^2d\mu_{s,t}\le e^{-\delta_0 t}\int_{L_s}|H_s|^2d\mu_s\le V_0\epsilon_0^2e^{-\delta_0 t},\quad t\in[0,T].\tag{6.18}$$

Note that $L_{s,t}\in\mathcal{A}(\frac{1}{3}\kappa_0, r_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}})$ for $t\in[0,T]$, by Lemma 3.7 there is a constant $C_1=C_1(n,\Lambda_0,K_2)$ such that

$$|\nabla A_{s,t}|\le C_1(n,\Lambda_0,K_2,\tau),\quad t\in[\tau,T].\tag{6.19}$$

Here we can choose $\tau=\tau(n,\Lambda_0,K_1)$ in Lemma 6.7. Thus, by Lemma 3.5 and (6.18)(6.19) we have

$$|H_{s,t}|\le\Big(\sqrt{\frac{3}{\kappa_0}}+C_1\Big)V_0^{\frac{1}{n+2}}\epsilon_0^{\frac{2}{n+2}}e^{-\frac{\delta_0}{n+2}t},\quad t\in[\tau,T].\tag{6.20}$$

where we have used the fact that L_t is $\frac{\kappa}{3}$-noncollapsed on the scale r_0 and $V_0\epsilon_0^2\le r_0^{n+2}$ if ϵ_0 is small enough. Thus, if ϵ_0 is small such that $\Big(\sqrt{\frac{3}{\kappa_0}}+C_1\Big)V_0^{\frac{1}{n+2}}\epsilon_0^{\frac{1}{n+2}}\le 1$, then we have

$$|H_{s,t}|\le\epsilon_0^{\frac{1}{n+2}}e^{-\frac{\delta_0}{n+2}t},\quad t\in[\tau,T].$$

(b). By Lemma 3.7 there exist some constants $C_k=C_k(n,\Lambda_0,K_{k+1})$ such that

$$|\nabla^k A_{s,t}|\le C_k(n,\Lambda_0,K_{k+1},\tau),\quad t\in[\tau,T].\tag{6.21}$$

By Lemma 3.7 and Property (a), we have

$$\int_{L_{s,t}}|\nabla^2 H_{s,t}|^2d\mu_{s,t}\le\int_{L_{s,t}}|H_{s,t}||\nabla^4H_{s,t}|d\mu_{s,t}\le V_0C_4\epsilon_0^{\frac{1}{n+2}}e^{-\frac{\delta_0}{n+2}t},\quad t\in[\tau,T],$$

where we used the fact that $\mathrm{Vol}(L_{s,t})$ is decreasing along the flow. Thus, by Lemma 3.5 we have

$$|\nabla^2 H_{s,t}| \le \Big(\sqrt{\frac{3}{\kappa_0}}+C_3\Big)C_4^{\frac{1}{n+2}}V_0^{\frac{1}{n+2}}\epsilon_0^{\frac{1}{(n+2)^2}}e^{-\frac{\delta_0}{(n+2)^2}t}, \quad t\in[\tau,T]. \tag{6.22}$$

Recall that by Lemma 2.3 $|A|$ satisfies the inequality

$$\frac{\partial}{\partial t}|A| \le |\nabla^2 H| + c(n)|A|^2|H| + |\bar{R}m||H|. \tag{6.23}$$

Thus, by Lemma 6.7, (6.22)(6.23) and (a) we have

$$\begin{aligned}|A_{s,t}| &\le |A_{s,\tau}| + \int_\tau^t |\nabla^2 H_{s,t}| + (K_0+|A|^2)|H_{s,t}| \\ &\le 2\Lambda_0 + \Big(\sqrt{\frac{3}{\kappa_0}}+C_3\Big)C_4^{\frac{1}{n+2}}V_0^{\frac{1}{n+2}}\epsilon_0^{\frac{1}{(n+2)^2}}\frac{(n+2)^2}{\delta_0} \\ &\quad +(K_0+36\Lambda_0^2)\epsilon_0^{\frac{1}{n+2}}\frac{n+2}{\delta_0} \\ &\le 3\Lambda_0, \end{aligned} \tag{6.24}$$

if we choose ϵ_0 sufficiently small.

(c). By (3.14), Lemma 6.7 Property $(a)(b)$ we have

$$\begin{aligned}E(t) &\le \int_0^\tau \max_L(|A||H|+|H|^2)\,ds + \int_\tau^t \max_L(|A||H|+|H|^2)\,ds \\ &\le 4\Lambda_0\epsilon_0\tau + 4\epsilon_0^2\tau + 3\Lambda_0\epsilon_0^{\frac{1}{n+2}}\frac{n+2}{\delta_0} + \epsilon_0^{\frac{2}{n+2}}\frac{n+2}{2\delta_0} \\ &\le \frac{1}{n+1}\log\frac{3}{2}, \quad t\in[0,T],\end{aligned}$$

where ϵ_0 is small enough. Thus, by Lemma 3.4 $L_{s,t}$ is $\frac{2}{3}\kappa_0$-noncollapsed on the scale r_0 for $t\in[0,T]$.

□

Now we can finish the proof of Theorem 1.4.

Proof of Theorem 1.4. . Suppose that $L_s\in\mathcal{A}(\kappa_0,r_0,\Lambda_0,\epsilon_0)$ for any positive constants κ_0, r_0, Λ_0 and small ϵ_0 which will be chosen later. Define

$$t_0 = \sup\Big\{t>0 \;\Big|\; L_{s,\xi}\in\mathcal{A}(\frac{1}{3}\kappa_0, r_0, 6\Lambda_0, 2\epsilon_0^{\frac{1}{n+2}}),\ \ \xi\in[0,t)\Big\}.$$

Suppose that $t_0<+\infty$. By Lemma 6.8, there exists $\epsilon_0=\epsilon_0(\kappa_0,r_0,\Lambda_0,n,K_5,V_0)>0$ such that $L_{s,t}\in\mathcal{A}(\frac{2}{3}\kappa_0,r_0,3\Lambda_0,\epsilon_0^{\frac{1}{n+2}})$ for all $t\in[0,t_0)$. Moreover, by Lemma 6.8 again the solution L_t can be extended to $[0,t_0+\delta]$ such that $L_{s,t}\in\mathcal{A}(\frac{1}{3}\kappa_0,r_0,6\Lambda_0,2\epsilon_0^{\frac{1}{n+2}})$, which contradicts the definition of t_0. Thus, $t_0=+\infty$ and

$$L_{s,t}\in\mathcal{A}(\frac{1}{3}\kappa_0,r_0,6\Lambda_0,2\epsilon_0^{\frac{1}{n+2}}), \quad t\in[0,\infty).$$

By Lemma 6.8 the mean curvature vector will decay exponentially to zero and the flow will converge to a smooth minimal Lagrangian submanifold. The theorem is proved.

□

7 Examples

In this section, we give some examples of minimal Lagrangian manifolds where Theorem 1.1 and Theorem 1.4 can be applied. However, to the author's knowledge, there is no examples of strictly hamiltonian stable minimal Lagrangian submanifold in Kähler-Einstein manifolds with positive scalar curvature.

Example 1: (cf. [8]) Let M_1, M_2 be closed Riemann surfaces with hyperbolic metrics g_1, g_2 respectively. Then $(M_1, g_1) \times (M_2, g_2)$ is a Kähler-Einstein surface of negative scalar curvature. Suppose that Σ be a closed surface with $\chi(\Sigma) = p_1\chi(M_1) = p_2\chi(M_2)$ where p_1, p_2 are positive constants and the map

$$f = (f_1, f_2) : \Sigma \to (M_1, g_1) \times (M_2, g_2)$$

satisfies $\deg f_1 = p_1, \ \deg f_2 = -p_2$ or $\deg f_1 = -p_1, \ \deg f_2 = p_2$. Then there exists a unique minimal Lagrangian surface L_0 in the homotopy class f. By Theorem 1.1, for any small Lagrangian perturbation of L_0 as the initial data, the mean curvature flow will exist for all time and converge exponentially to L_0.

Example 2: (cf. [14][15]) Consider the Clifford torus

$$\mathbb{T}^n = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n \mid |z_0| = |z_1| = \cdots = |z_n|\}.$$

It is proved in [14] that the Clifford torus is hamiltonian stable and the first eigenvalue of the Laplacian is $\lambda_1 = \frac{\bar{R}}{2n}$. By [15] the first eigenspace is spanned by the following functions restricted to the torus:

$$\mathrm{Re}(z_i), \ \mathrm{Im}(z_i), \ \mathrm{Re}(z_i\bar{z}_j), \ \mathrm{Im}(z_i\bar{z}_j) \tag{7.1}$$

for $0 \leq i \neq j \leq n$. Thus, if the initial data is any small hamiltonian deformation of $\mathbb{T}^n$ generated by a vector field $X = J\nabla f$ where f is not in the space spanned by (7.1), the mean curvature flow will exists for all time and deform it exponentially to a Clifford torus up to congruence by Theorem 1.4.

More generally, we have the following example where Theorem 1.4 can be applied:

Example 3: (cf. [16]) Let G be a compact semisimple Lie group, $\mathfrak{g}$ its Lie algebra, $(,)$ an Ad_G-invariant inner product on $\mathfrak{g}$, and M an adjoint orbit in $\mathfrak{g}$ with the associate 2-form equal to the canonical symplectic form. If $(M, (,))$ is Kähler-Einstein with positive scalar curvature and $L \subset M$ is a closed minimal Lagrangian submanifold, then $\lambda_1 = \frac{\bar{R}}{2n}$ and L is hamiltonian stable. Morover, all of the coordinate functions of $L \to \mathfrak{g}$ are in the first eigenspace of L. Thus, as in Example 2, Theorem 1.4 can be applied in this situation.

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